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A THEORY OF LINEAR DISTANCE AND ANGLE.

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Introduction.

1. In a recent article [1] we developed for the plane a theory of distance and angle such that points equally distant from a fixed point lie on a line and lines making a given angle with a fixed line pass through a point. On account of this property we have called this distance *linear*. In the present paper we extend this theory to higher dimensions. Because of the increased complexity, the synthetic method of the previous discussion cannot be used here and since we know none better we have adopted that of Grassmann. In the first part of the paper we have shown how the extensive quantities of Grassmann can be regarded as matrices and the progressive and regressive multiplication interpreted as simple operations performed upon these matrices. In this way we develop as much of the Grassmann analysis as is needed for our purpose. We then determine for any two spaces R, R' of the same dimension, a distance or angle $\overline{R\,R'}$ having the property that if this invariant is constant and either of the spaces fixed, the other satisfies a linear relation and such that for three spaces R, R', R'' of a pencil

$$\overline{R\,R'} + \overline{R'\,R''} + \overline{R''\,R} = 0.$$

Any distance between points that has these properties is expressible in terms of a hyperplane and a linear line complex. The plane is the locus of infinitely distant points and the complex the locus of minimal lines. If the complex does not degenerate, the hyperplane and line complex in n dimensions determine a point and $n-2$ other complexes forming altogether n elements which we use for a reference system. This system of elements forms a group under outer multiplication in the sense that any product of these elements is equal to a numerical multiple of a third one. In terms of this fundamental system we define the angle between any two spaces. Each of the complexes of the fundamental system is an infinite locus for spaces complimentary to it. The entire system is invariant under a group of collineations of the same order as the Euclidean group of motions. Degenerate cases are obtained by taking sections having a special relation to the fundamental system.

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[1] An Algebra of Plane Projective Geometry, Proceedings of the American Academy of Arts and Sciences, Vol. 47, p. 737.

Matrices in Three Dimensions.

2. **Progressive Matrices.** We represent a point A in three dimensions by a set of four homogeneous coordinates a_i. These coordinates determine a matrix

$$A = \| a_1 \, a_2 \, a_3 \, a_4 \| = \| a_i \|$$

which may be used to represent the point. Two matrices of this kind will be called equal when their corresponding elements are equal. The matrix is zero if all its elements are zero. If $a_i = k \, b_i$ we shall write

$$A = k \, B.$$

In this case the matrices A and B represent the same point but with different magnitudes. A linear function of A and B is defined by the matrix

$$\lambda \, A + \mu \, B = \| \lambda \, a_i + \mu \, b_i \|.$$

In a similar manner we define any linear function of points or matrices A, B, C, etc. If the result does not vanish it represents a point in the space determined by A, B, C, etc. If it vanishes and the coefficients are not all zero those points lie in a lower space than a like number of points usually determine.

The coordinates of the line joining A and B are proportional to the two-rowed determinants in the matrix

$$[A \, B] = \left\| \begin{matrix} a_1 \, a_2 \, a_3 \, a_4 \\ b_1 \, b_2 \, b_3 \, b_4 \end{matrix} \right\| = \left\| \begin{vmatrix} a_i \, a_k \\ b_i \, b_k \end{vmatrix} \right\|.$$

We shall call the elements of this matrix the two-rowed determinants

$$\begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix}.$$

[This is not in conformity with the usual definition which makes element equivalent to coordinate a_i or b_i but is the only definition which has a value in the present discussion.] The matrix is zero when all its elements are zero. In that case the points A and B have proportional coordinates and hence coincide. If the matrix is not zero it represents the line $A\,B$ in the sense that from the matrix can be obtained the coordinates of the line. Conversely if the line is given a matrix can be formed by taking any two points on the line. Different matrices representing the same line are multiples of any one. For if A, B and P, Q are pairs of distinct points on the line

$$P = \lambda_1 A + \lambda_2 B,$$
$$Q = \mu_1 A + \mu_2 B,$$

and

$$[P\,Q] = \begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} [A\,B].$$

Thus a two-rowed matrix in addition to representing a line, has a definite size.

The matrix $[A\,B]$ is in reality a set of six determinants

$$\begin{vmatrix} a_i & a_k \\ b_i & b_k \end{vmatrix}$$

taken in some definite order. It can then be considered as a one-rowed matrix of six terms

$$[A\,B] = \| a_i\, b_k - a_k\, b_i \|.$$

The sum of two matrices $[A\,B]$ and $[C\,D]$ is then a complex matrix, each element of which is the sum of corresponding elements in $[A\,B]$ and $[C\,D]$. In general this sum cannot be represented as a single two-rowed matrix, just as the sum of corresponding Plücker coordinates of two lines are not in general coordinates of a line. For analytical purposes we express this sum by simply writing the two matrices with an addition sign between them. If, however, the lines $A\,B$ and $C\,D$ intersect in a point P, we can find points Q and R on those lines such that

$$[A\,B] = [P\,Q],$$
$$[C\,D] = [P\,R].$$

Then

$$[A\,B] + [C\,D] = [P\,Q] + [P\,R] = \left\| \begin{matrix} p_i & p_k \\ q_i + r_i & q_k + r_k \end{matrix} \right\| = [P(Q + R)].$$

We can consider $[A\,B]$ as a product of A and B. For

$$[A(B + C)] = [A\,B] + [A\,C]$$

as we have just seen in the case of $[P(Q + R)]$. The process of multiplication consists in placing the second matrix under the matrix A

to form a two rowed matrix $[A\ B]$. We shall call this the *progressive* product. From the definition it is evident that

$$[A\ B] = -[B\ A]$$

and

$$[A\ A] = 0.$$

3. If the points A, B, C are not collinear, the coordinates of the plane $A\ B\ C$ are proportional to the three rowed determinants in the matrix

$$[A\ B\ C] = \left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix}\right\|.$$

This matrix represents the plane in the sense that from it we can determine the coordinates of the plane. Conversely to represent any plane as a matrix we take three non-collinear points of the plane and form the matrix from them. The elements of such a matrix are the three-rowed determinants belonging to it. In reality we are considering this as a one-rowed matrix of four terms (equal to the three-rowed determinants in $[A\ B\ C]$) arranged in some definite order. Two matrices of this kind will be called equal if corresponding elements are equal and are added by adding corresponding elements.

If P, Q, R are any three points of the plane determined by A, B, C,

$$\begin{aligned} P &= \lambda_1 A + \lambda_2 B + \lambda_3 C, \\ Q &= \mu_1 A + \mu_2 B + \mu_3 C, \\ R &= \nu_1 A + \nu_2 B + \nu_3 C. \end{aligned}$$

Consequently

$$[P\ Q\ R] = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} [A\ B\ C].$$

Thus a matrix $[P\ Q\ R]$ in addition to representing a plane has a definite size. The vanishing of $[P\ Q\ R]$ signifies that P, Q, R lie on a line.

The matrix $[A\ B\ C]$ can be regarded as a product of $[A\ B]$ and C, A and $[B\ C]$ or of A, B and C, the process of multiplication consisting always of placing the first matrix at the top and the others in order under it to form a single matrix.[2]

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[2] In this multiplication each matrix must have four columns. If instead of $[A\ B]$ we have a complex the operation must be performed distributively on each two-rowed matrix of the sum. For purposes of addition we regard our quantities as matrices of one row but for purposes of multiplication as matrices or sums of matrices of four columns.

From this definition it is evident that

$$\begin{aligned} [A\cdot B\,C] &= [A\,B\cdot C] = [A\,B\,C], \\ [A\,B\,(C+D)] &= [A\,B\,C] + [A\,B\,D], \\ [A\,B\,C] &= -[A\,C\,B] = [C\,A\,B]. \end{aligned}$$

The sum of any number of three-rowed matrices can be expressed as a single three-rowed matrix $[P\,Q\,R]$. In fact let $A_1\,B_1\,C_1$ and $A_2\,B_2\,C_2$ cut in a line $P\,Q$. Then

$$\begin{aligned} [A_1\,B_1\,C_1] &= [P\,Q\,R_1], \\ [A_2\,B_2\,C_2] &= [P\,Q\,R_2]. \end{aligned}$$

Hence

$$\begin{aligned} [A_1\,B_1\,C_1] + [A_2\,B_2\,C_2] &= [P\,Q\,R_1] + [P\,Q\,R_2] \\ &= [P\,Q\,(R_1+R_2)] = [P\,Q\,R], \end{aligned}$$

where

$$R = R_1 + R_2.$$

From four points A, B, C, D we can form a four rowed matrix or determinant

$$(A\,B\,C\,D) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

This matrix has only one element and hence we write it as a determinant. A matrix of one element is analytically equivalent to a number. We use the parentheses to indicate this fact. Square brackets are used to represent matrices which do not reduce to numbers. The vanishing of $(A\;B\;C\;D)$ is the condition that the four points lie in a plane.

The quantity $(A\;B\;C\;D)$ can be regarded as a product in a number of ways. From the definition it is evident that

$$(A\,B\,C\,D) = (A\cdot B\,C\,D) = (A\,B\cdot C\,D) = -(A\,B\,D\,C).$$

4. **Regressive Matrices.** We can consider space as generated by planes as well as by points. If its coordinates are α_i, a plane α is then represented by a matrix

$$\alpha = \|\,\alpha_1\;\alpha_2\;\alpha_3\;\alpha_4\,\|.$$

The same plane may be represented by a matrix $[A\;B\;C]$. Then the

coordinates a_i are proportional to the coefficients of x_i in the determinant $|A\ B\ C\ X|$.[3]
If a_i is equal to the coefficients of x_i in that determinant we shall write

$$\alpha = [A\ B\ C].$$

Thus a three-rowed matrix is for our purpose equivalent to a one-rowed matrix in contragredient variables.

The line of intersection of two planes α and β can be represented by a matrix

$$[\alpha\,\beta] = \left\|\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{matrix}\right\|.$$

The coordinates of the line are here

$$q_{ik} = \begin{vmatrix} \alpha_i & \alpha_k \\ \beta_i & \beta_k \end{vmatrix}.$$

If the same line is the join of two points A and B we know from analytical geometry that the coordinates q_{ik} are proportional to the coefficients of the minors $\begin{vmatrix} x_i & x_k \\ y_i & y_k \end{vmatrix}$ in the determinant $|\,A\ \ B\ X\ \ Y\,|$. If q_{ik} is equal to the coefficient of $|\,x_i\ y_k\,|$ in the determinant we shall write

$$[\alpha\,\beta] = [A\ B].$$

This amounts to saying that in the determinant $[A\ B\ \alpha\,\beta\,]$, each minor in the first two rows is then equal to its algebraic compliment (coefficient in the expansion of the determinant).

Similarly we represent the point of intersection of three planes by a matrix $[\alpha\,\beta\,\gamma]$. The coordinates a_i of this point A are proportional to the coefficients of ξ_i in the determinant $[\xi\ \alpha\,\beta\,\gamma]$. In particular if a_i is equal to the coefficient of ξ_i in that determinant we write

$$A = [\alpha\,\beta\,\gamma].$$

In this case each term of the first row in the determinant $(A\ \alpha\,\beta\,\gamma)$ is equal to its algebraic compliment.

There is a determinant $[\alpha\,\beta\,\gamma\ \delta]$ of four planes just as of four points. These quantities $[\alpha\,\beta]$, $[\alpha\,\beta\,\gamma]$, $(\alpha\,\beta\,\gamma\ \delta)$ can be regarded as products formed according to the same laws as the products of points. These products of matrices expressed in plane coordinates we shall call *regressive*.

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[3] It is to be observed that here X is written last. If we take the coefficients of X_i in the determinant $|X\ A\ B\ C|$ they will have different signs from the coefficients used here.

Matrices in Hyperspace.

5. We shall call the order of a space the number of homogeneous coordinates of a point in that space.[4] Thus a point is of order one, a line of order two, etc. A space of order n can be generated either by points or by hyperplanes of order n–1. A space R of order $r < n$ can be determined by a set of r points A, giving rise to a matrix.

$$R \equiv \left\| \begin{matrix} a_{11} & a_{12} \ldots & \ldots a_{1n} \\ a_{21} & a_{22} \ldots & \ldots a_{2n} \\ \ldots & \ldots & \ldots \\ a_{r1} & a_{r2} \ldots & \ldots a_{rn} \end{matrix} \right\| = [A_1 A_2 \ldots A_r].$$

This matrix represents the space R in the sense that from the matrix can be determined the coordinates of the space. The same space R can be determined as the intersection of n–r hyperplanes α_i determining a matrix

$$\left\| \begin{matrix} \alpha_{r+1,\,1} & \alpha_{r+1,\,2} \ldots & \ldots \alpha_{r+1,\,n} \\ \alpha_{r+2,\,1} & \alpha_{r+2,\,2} \ldots & \ldots \alpha_{r+2,\,n} \\ \ldots & \ldots & \ldots \\ \alpha_{n\;1} & \alpha_{n,\,2} \ldots & \ldots \alpha_{n,\,n} \end{matrix} \right\| = [\alpha_{r+1}\ \alpha_{r+2} \ldots \alpha_n].$$

The condition that these matrices represent the same space is that in the determinant

$$\left| \begin{matrix} a_{11} & a_{12} \ldots & \ldots a_{1n} \\ \ldots & \ldots & \ldots \\ a_{r1} & a_{r2} \ldots & \ldots a_{rn} \\ \alpha_{r+1,\,1} & \alpha_{r+1,\,2} \ldots & \ldots \alpha_{r+1,\,n} \\ \ldots & \ldots & \ldots \\ \alpha_{n,\,1} & \alpha_{n2} \ldots & \ldots \alpha_{nn} \end{matrix} \right|$$

the minors in the first r rows be proportional to their algebraic compliments.[5] If in the determinant each minor of the first r rows is *equal* to its algebraic compliment we shall write

$$[A_1\ A_2 \ldots A_r] = [\alpha_{r+1}\ \alpha_{r+2} \ldots \alpha_n].$$

The r-rowed determinants of a matrix of r rows we call the elements of the matrix. To add such matrices we add corresponding elements. If there exists a matrix of n columns whose elements are the corresponding sums, it represents the sum of the given matrices. If no such matrix exists the sum is complex. In that case we write the result as an algebraic sum and do not attempt to express it as a single matrix. If some of the matrices are expressed in point coordinates, the others in hyperplane coordinates we replace those of one kind by their dual forms or at least imagine them so replaced. This amounts to adding elements of the former to their algebraic compliments in the latter and considering the result as a term of the first kind.

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[4] Cf. Whitehead, Universal Algebra, page 177.

[5] Cf. Bertini, Geometria Projettiva, p. 33.

6. A matrix in which the number of rows is equal to or less than the number of columns can be regarded as a product. If the matrix is expressed in point coordinates we call the product progressive, if in hyperplanar coordinates regressive. To multiply two such matrices (of the same kind) the sum of whose rows is equal to or less than n, we write the second matrix under the first to form a single matrix. If one of the factors is complex we apply the process distributively to the separate matrices of the sum. From the definition it is evident that such products are distributive and associative and that the interchange of two points or hyperplanes (according as the product is progressive or regressive) changes the sign of the result.

If a matrix of r rows vanishes, all the minors of order r in that matrix are zero. There is then a linear relation between the rows of the matrix.[6] If the matrix represents a progressive product of r points there is a linear relation between the points of that product and they therefore lie in a space of order less than r. If the matrix represents a regressive product of r hyperplanes they satisfy a linear relation and therefore intersect in a space of order greater than $n-r$. If the matrix is not zero the progressive form represents the space containing its factors and the regressive the space common to its factors.

The most general product is the result of a succession of operations each consisting of multiplying two factors. If the total number of rows in two matrices of the same kind (progressive or regressive) is less than n, the two are multiplied together according to the rule already given. If the total number of rows is greater than n, the product as previously defined gives a matrix of more rows than columns. For such a matrix we have no interpretation. In that case we replace each factor by its equivalent in contragredient variables. The total number of rows in the new product is less than n and we form the product by the previous method. If the total number of rows is equal to n the result is the same whether the matrices are taken in point or hyperplane coordinates. If the matrices are of different kinds we replace one of them by its contragredient form in such a way that the new matrices have a sum of rows equal to or less than n. Thus in every case of a product there is a definite result that has a meaning. We call this the *product* of those factors.

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[6] Böcher, Introduction to Higher Algebra, p. 36.

7. **Reduction formulae.** We have just found that in expressing the product of two matrices when the sum of the rows is greater than n, we must change to contragredient forms. We shall now derive certain reduction formulae by which we obtain the same results without that change. For this purpose let

$$\begin{aligned} [A_1 \;\; A_2 \ldots . A_p] &= [\alpha_{p+1}, \;\; \alpha_{p+2} \ldots . \alpha_n] \\ [B_1 \;\; B_2 \ldots . B_q] &= [\beta_{q+1}, \;\; \beta_{q+2} \ldots . \beta_n]. \end{aligned}$$

We shall now prove that in the determinant

$$\Delta \equiv \left| \begin{array}{llllll} a_{11} & a_{12} & \ldots . a_{1n} & \pm a_{11} & \pm a_{12} & \ldots . \pm a_{1n} \\ \ldots & \ldots & \ldots\ldots . & \ldots . & \ldots . . & \ldots\ldots\ldots \\ a_{p1} & a_{p2} & \ldots . a_{pn} & \pm a_{p1} & \pm a_{p2} & \ldots . \pm a_{pn} \\ \alpha_{p+1,1} & \alpha_{p+1,2} & \ldots . \alpha_{p+1,n} & 0 & 0 & \ldots . 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \alpha_{n,1} & \alpha_{n,2} & \ldots . \alpha_{n,n} & 0 & 0 & \ldots . 0 \\ 0 & 0 & \ldots . 0 & b_{11} & b_{12} & \ldots . b_{1n} \\ 0 & 0 & \ldots . 0 & b_{q1} & b_{q2} & \ldots . b_{qn} \\ \mp \beta_{q+1,1} & \mp \beta_{q+1,2} & \ldots . \mp \beta_{q+1,n} & \beta_{q+1,1} & \beta_{q+1,2} & \ldots . \beta_{q+1,n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \mp \beta_{n,1} & \mp \beta_{n,2} & \ldots . \mp \beta_{n,n} & \beta_{n,1} & \beta_{n,2} & \ldots . \beta_{n,n} \end{array} \right|$$

each minor from the $n - p$ rows of α's and $n - q$ rows of β's is equal to its algebraic compliment. To prove this we first show that if such a minor M contains a product of a minor A whose order is $n - p$ in the α's by a minor B of order $n - q$ in the β's, then the algebraic complement of M contains a product of minors respectively equal to A and B. Since A is contained in the principal minor $| a_{11} \ldots . \alpha_{nn}|$, if B is in its complement $| b_{11} \ldots . \beta_{nn}|$, the result is obvious. For the algebraic complements of A and B respectively in those principal minors are the terms required. If B is not contained entirely in the principal minor, there is a minor B', in $| b_{11} \ldots . \beta_{nn} |$ containing the same letters as B and in the same order (but having perhaps different signs). In the algebraic complement of a minor M' containing A, B' is then a term $C\, D = A\, B'$ where C is a minor of p rows of a's and D a minor of q rows of b's. If now we permute the columns contained in B' but not in B with their correspondents of the same suffix, this term will pass over into a term of the complement of M. In this process C and D are not changed for each a_{ik} in C is either not changed at all or replaced by the same letter and no letter of D can be in a column so moved. Furthermore the sign is correct for there are as many minus columns introduced as interchanges made. The same argument shows that for every product of minors in the algebraic complement of M there is an equal product in M. Therefore M is equal to its algebraic complement.

Suppose now $p + q \geqq n$. Then

$$n - p + n - q \leqq n.$$

We expand the determinant Δ in terms of minors of the nth order taken from the first n and the last n columns. The *part* of the expansion which contains all the α's and β's in the minor from the first n rows is

$$\Delta_1 = \Sigma\,(A_1\ A_2 \ldots . A_{n-q}\ B_1\ B_2 \ldots . B_q)\ (A_{n-q+1} \ldots . A_p\ \alpha_{p+1} \ldots .. \alpha_n \\ \beta_{q+1}\ \ldots . \beta_n)$$

the summation being for every combination of $n - q$ A's in the first factor with the remaining $p + q - n$ in the second factor so arranged that the two groups in the order written constitute a positive permutation of A_1 to A_p. The form of this expression is evident since the B's cannot occur in the same factor with α's and β's (the other factor then containing a row of zeros). The sign of the term written is positive since it is obtained as a product of principal minors given by moving q rows of B's past $n - p$ rows of α's and $p + q - n$ rows of A's, interchanging first n and last n columns and changing $n - q$ minus signs. The result should therefore have a sign

$$(-1)^{q(n-p+p+q-n)+n^2+n-q} = 1.$$

The signs of the other terms then follow, since any positive rearrangement of A's should not change the sign of the term.

Now in the expression of Δ each minor formed of $n - p$ rows of α's and $n - q$ rows of β's is equal to its coefficient. Furthermore Δ_1 contains all of the terms in Δ given by such minors taken from the first n columns. Therefore in Δ_1 each minor of the matrix $[\alpha_{p+1} \ldots \alpha_n\ \beta_{q+1} \ldots \beta_n]$ is equal to its coefficient. These coefficients constitute the matrix

$$\Sigma\,(A_1 \ldots A_{n-q}\ B_1 \ldots B_q)\ [A_{n-q+1} \ldots . A_p]$$

which is therefore equal to the former. We can write this result in the form [7]

$$[\alpha_{p+1}\ldots\alpha_n\ \beta_{q+1}\ldots\beta_n] = [A_1\ A_2\ldots A_p\ B_1\ B_2\ldots B_q] = \Sigma\,(C\ B_1\ldots B_q)\ D \qquad (1)$$

C being the product of any combination of $n - q$ of A's and D the product of the others such that

$$[A_1\ A_2\ldots A_p] = [C\ D]$$

If $p + q < n$ we take the part of Δ which contains all the A's and B's in one n-rowed minor. The result is

$$\Delta_2 = \Sigma(\alpha_{p+1}\ldots\alpha_n\ \beta_{n-p+1}\ldots\beta_n)[A_1\ldots A_p\ B_1\ldots B_q\ \beta_{q+1}\ldots\beta_{n-p}].$$

Hence we have

$$[A_1\ldots A_p\ B_1\ldots B_q] = [\alpha_{p+1}\ldots\alpha_n\ \beta_{q+1}\ldots\beta_n] = \Sigma\,(\alpha_{p+1}\ldots\alpha_n\,\delta)\ \gamma \qquad (2)$$

where δ is any combination of p B's and γ the remaining ones so arranged that

$$[\beta_{q+1}\ldots\beta_n] = [\,\gamma\ \delta\,].$$

If instead of the determinant Δ we use the determinant

$$\Delta' \equiv \begin{vmatrix} a_{11} & a_{12}\ldots\ldots\ldots a_{1n} & 0 & 0\ldots\ldots\ldots 0\ldots \\ \ldots & \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \\ a_{p1} & a_{p2}\ldots\ldots\ldots a_{pn} & 0 & 0\ldots\ldots\ldots 0\ldots \\ \alpha_{p+1,\,1} & \alpha_{p+1,\,2}\ldots\alpha_{p+1,\,n} & \pm\alpha_{p+1,\,1} & \pm\alpha_{p+1,\,2}\ldots\pm\alpha_{p+1,\,n} \\ \ldots & \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \\ \alpha_{n,\,1} & \alpha_{n,\,2}\ldots\ldots\ldots\alpha_{n\,n} & \pm\alpha_{n1} & \alpha_{n2}\ldots\ldots\alpha_{nn}\ldots \\ \mp b_{11} & \mp b_{12}\ldots\ldots\mp b_{1n} & b_{11} & b_{12}\ldots\ldots b_{1n}\ldots \\ \ldots & \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \\ \mp b_{q1} & \mp b_{q2}\ldots\ldots\mp b_{q\cdot n} & b_{q1} & b_{q2}\ldots\ldots b_{qn}\ldots \\ 0 & 0\ldots\ldots\ldots 0 & \beta_{q+1,\,1} & \beta_{q+1,\,2}\ldots\beta_{q+1,\,n} \\ \ldots & \ldots\ldots\ldots\ldots & \ldots & \ldots\ldots\ldots\ldots \\ 0 & 0\ldots\ldots\ldots\ldots 0 & \beta_{n1} & \beta_{n2}\ldots\ldots\beta_{nn}\ldots \end{vmatrix}$$

when $p + q > n$, we obtain the expression in the form

$$[A_1\,A_2\ldots A_p\,B_1\,B_2\ldots B_p] = \Sigma\,(A_1\,A_2\ldots A_p\,D)\ C\ . \qquad . \quad (3)$$

where D is any combination of $n-p$ letters B and C the remaining ones so arranged that

$$[B_1\,B_2\ldots B_q] = [C\ D].$$

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[7] Grassmann, Gesamelte Werke, Vol. I, p. 83; Whitehead, Universal Algebra, p. 188; H. B. Philips, Proceedings of the American Academy of Arts and Sciences, Vol. 46, p. 369. In this last article the formula obtained may have a different sign from the one here given.

When $p + q < n$ this determinant gives

$$[\alpha_{p+1} \ldots \alpha_n \, \beta_{q+1} \ldots \beta_n] = \Sigma \, (\gamma \, \beta_{q+1} \ldots \beta_n) \, \delta \quad . \quad . \qquad (4)$$

where γ is any combination of q α's and δ the remaining ones so arranged that

$$[\alpha_{p+1} \ldots \alpha_n] = [\gamma \, \delta].$$

Symbolic notation.

8. The determinants in the matrix representing a space S are the coordinates s_i of S. Of σ is a space complimentary to S, we consider it as represented by a matrix of the same kind as S. It has then a like number of coordniates σ_i (algebraic compliments of s_i in the determinant $|s\sigma|$). Then

$$(S \, \sigma) = \Sigma \, s_i \, \sigma_i.$$

This is a linear function of the coordinates s_i and by a proper choice of σ (perhaps complex) can be made any linear function of those coordinates. To obtain a bilinear function of the coordinates r_i, s_k of two spaces R and S we take matrices ρ and σ complimentary to R and S. Then

$$(R \, \rho) \; (S \, \sigma) = \Sigma \, \rho_i \, \sigma_k \, r_i \, s_k \, . \quad . \quad . \quad . \qquad (5)$$

In order to obtain the most general bilinear function

$$\Sigma \, a_{ik} \, r_i \, s_k$$

we consider the above as a symbolic representation in which $\rho_i \, \sigma_k$ is to be replaced by a_{ik}. Thus $(R \, \rho)$ $(S \, \sigma)$ represents symbolically any bilinear function of the coordinates r_i, S_k. Any linear relation connecting the symbolic quantities $(R \, \rho)$ $(S \, \sigma)$ will be satisfied by the bilinear functions $\Sigma \, a_{ik} \, r_i \, s_k$. This is the symbolic representation so much used by Clebsch.

We can consider $(R \, \rho)$ $(S \, \sigma)$ as resulting from an expression $\rho \, \sigma$ by operating on the first factor with R and on the second with S. This product $\rho \, \sigma$ is the dyadic of Gibbs.[8] It may be considered as a distributive product of ρ and σ. It is called the indeterminate[9] product. In it the order of factors must be preserved. In fact there is no general functional relation between $\rho \, \sigma$ and $\sigma \, \rho$. The dyadic

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[8] Vector Analysis, Gibbs-Wilson, page 265.

[9] Cf. H. B. Phillips, loc. cit.

$\rho\,\sigma$ represents a transformation which changes a space R complimentary to ρ into a space $(R\,\rho)\,\sigma$ which is given by the locus of S in

$$(R\,\rho)\,(S\,\sigma) = 0.$$

Linear Distance and Angle in Three Dimensions.

9. **Linear distance between two points.** We define the distance between two points A, B as such a function $\overline{A\,B}$ of their coordinates that (1) if one is fixed the other lies in a plane, and (2) for points A, B, C on a line

$$AB + BC + CA = 0\,. \tag{6}$$

From the first condition the distance must be of the form

$$\overline{A\,B} = \frac{F_1\,(A,B)}{F_2\,(A,B)} \tag{7}$$

where F_1 and F_2 are bilinear functions of A and B. Putting A, B and C equal in the second condition we get

$$\overline{A\,A} = 0.$$

Hence

$$F_1\,(A_1\,A) = 0 \tag{8}$$

In this last equation replacing A by $A + B$ and cancelling the terms $F_1\,(A, A)$ and $F_1\,(B, B)$ we have

$$F_1\,(A, B) + F_1\,(B, A) = 0\,. \tag{9}$$

The numerator of $\overline{A\,B}$ must then change sign when we interchange A and B. In (6) putting $C = B$ we have

$$\overline{A\,B} + \overline{B\,A} = 0.$$

This shows that

$$F_2\,(A, B) = F_2\,(B, A)\,. \tag{10}$$

or the denominator of $\overline{AB}$ is symmetric in A and B. Let $C = A + B$. Then (6) becomes

$$\frac{F_1\,(A,B)}{F_2\,(A,B)} + \frac{F_1\,(B,A) + F_1\,(B,B)}{F_2\,(B,A) + F_2\,(B,B)} + \frac{F_1\,(A,A) + F_1\,(B,A)}{F_2\,(A,A) + F_2\,(B,A)} = 0.$$

Making use of (8), (9) and (10) this becomes

$$F_1\,(A,B)\,[F_2\,(A,A)\,F_2\,(B,B) - F_2\,(A,B)^2] = 0.$$

Then either

$$F_1(A, B) = 0$$

or

$$F_2(A, B)^2 = F_2(A, A)\,F_2(B, B).$$

This last equation shows that $F_2(A, B)$ factors into a function of A times a function of B. Calling this function $(\phi\,A)$, and writing

$$\mathrm{F}_1(A, B) = \tfrac{1}{2}\,[F_1(A, B) - F_1(B, A)] = (\alpha\,A)\,(\beta\,B) - (\alpha\,B)\,(\beta\,A),$$

we have

$$\overline{A\,B} = \frac{(\alpha\,A)\,(\beta\,B) - (\alpha\,B)\,(\beta\,A)}{(\phi\,A)\,(\phi\,B)}.$$

Using the identity (3) this takes the form [10]

$$\overline{A\,B} = \frac{(\alpha\,\beta\cdot A\,B)}{(\phi\,A)\,(\phi\,B)} = \frac{(q\cdot A\,B)}{(\phi\,A)\,(\phi\,B)},$$

where we put q in place of the two rowed matrix $(\alpha\;\beta]$.

10. **Angle between two planes**. We define the angle between two planes as such a function $\overline{\alpha\beta}$ of their coordinates that if the angle is given and one of the planes fixed, the other passes through a point and for three planes of a linear pencil

$$\overline{\alpha\beta} + \overline{\beta\gamma} + \overline{\gamma\,\alpha} = 0.$$

By the same argument as for the distance between two points we obtain for the angle

$$\overline{\alpha\beta} = \frac{(p\cdot\alpha\,\beta)}{(F\,\alpha)\,(F\,\beta)}$$

where p is a fixed complex and F a fixed point.

Distance is a relative invariant under the group of collineations that leave the complex q and the plane ϕ fixed. Similarly angle is a relative invariant under the group leaving p and F fixed. In order that fixed relations may exist between distances and angles we wish, if possible, these groups to be the same. We assume that the complex q does not degenerate into a line. Then the only complex and point determined by q and ϕ is the complex q itself and the polar point of ϕ with respect to it. Hence we have

$$p = q,$$
$$F = \lambda\,[\phi\,p].$$

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[10] We consider $(\boldsymbol{\alpha}\,\boldsymbol{\beta}.A\,B)$ as a regressive product $(\boldsymbol{\alpha}.\boldsymbol{\beta}.A\,B)$, in which we expand the product $(\boldsymbol{\beta}.A\,B)$ and then multiply by $\boldsymbol{\alpha}$.

We choose the unit angle such that $\lambda = 1$. Then

$$F = [\phi\, p].$$

also[11]

$$[F\, p] = [\phi\, p\,.\,p] = \tfrac{1}{2}\,(p.\, p)\,\phi = \phi$$

if we choose the magnitude of p such that

$$(p\, p) = 2.$$

The relations between ϕ and F are then symmetrical.

Our formulae are now

$$\overline{A\,B} = \frac{(p\,A\,B)}{(\phi\,A)\,(\phi\,B)} \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

$$\overline{\alpha\,\beta} = \frac{(p\cdot\alpha\,\beta)}{(F\,\alpha)\,(F\,\beta)} \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

with the condition that $F = [\phi\, p]$, $\phi = [F\, p]$ and $(p\,p) = 2$.

The ratio of two distances or of two angles, also the product of a distance and angle are invariant under the seven parameter group of collineations leaving the complex p and the plane ϕ fixed. If one of these transformations leaves a distance or angle unchanged it leaves all distances and angles unchanged. Those quantities then are invariant under a six parameter group. Any tetrahedron can therefore be transformed into an equal tetrahedron (one having equal length of sides) by a collineation leaving distance and angle invariant.

From the formula for the distance between two points, it is seen that distances along a line of the complex p are zero provided neither of the points lies on ϕ. The distance along a line of p to ϕ is indeterminate but along any other line it is infinite. Similarly the angle between two planes intersecting in a line of p but neither passing through F is zero. If one of the planes passes through F, the angle is indeterminate or infinite according as the other plane does or does not cut it in a line of p.

11. The locus of points y at a distance λ from the point x is a plane

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[11] The formula $[\phi\, p\cdot p] = \frac{1}{2}\,(p\,p)\,\phi$ can be proved as follows. Let

$$p = \alpha\,\beta + \gamma\,\delta.$$

Then $(p\,p) = 2(\alpha\,\beta\,\gamma\,\delta)$

and

$$\begin{aligned}[\phi\, p\cdot p] &= [\phi(\alpha\,\beta + \gamma\,\delta)\cdot(\alpha\,\beta + \gamma\,\delta)] \\ &= (\phi\,\alpha\,\beta\,\delta)\,\gamma - (\phi\,\alpha\,\beta\,\gamma)\,\delta + (\phi\,\alpha\,\delta\,\beta)\alpha - (\phi\,\gamma\,\delta\,\alpha)\,\beta \\ &= \phi\,(\alpha\,\beta\,\gamma\,\delta) = \tfrac{1}{2}\,(p\,p)\,\phi.\end{aligned}$$

ξ intersecting ϕ on the polar plane of x with respect to p. The correspondence between x and ξ is a correlation. From the equation

$$\overline{x\,y} = \frac{(p\,x\,y)}{(\phi\,x)\,(\phi\,y)} = \lambda,$$

or

$$\lambda\,(\phi\,x)\,(\phi\,y) - (p\,x\,y) = 0,$$

it is seen that the locus of y is

$$\xi = \lambda\,(\phi\,x)\,\phi - [p\,x] \qquad . \quad . \quad . \; (13)$$

Similarly the locus of planes η making a given angle λ with the plane ξ is a point such that the line connecting it to F passes through the polar point of ξ with respect to p. The locus of planes making with ξ an angle $-\lambda$ is

$$z = -\lambda\,(F\,\xi)\,F - [p\xi].$$

Substituting in this the value of ξ from (13) we get

$$z = \lambda\,(F\,p\,x)\,F - \lambda\,(\phi\,x)\,[p\,\phi] + [p\cdot p\,x]$$

since $(F\,\phi) = 0$, F being a point of ϕ. Using the conditions

$$[F\,p] = \phi, \quad [\phi\,p] = F, \quad \text{and} \quad [p\cdot p\,x] = \tfrac{1}{2}\,(p\,p)\,x = x$$

we get

$$z = x$$

Hence the correlations determined by a distance λ and by an angle $-\lambda$ are inverse. Now the correlation set up by an angle $-\lambda$ is inverse to that determined by an angle λ. Hence the equations

$$\overline{x\,y} = \lambda,$$

$$\overline{\xi\,n} = \lambda,$$

where x and ξ are given, y and η variable, set up the same correlation.

Through a correlation

$$\overline{x\,y} = \lambda$$

to x_1 and x_2 correspond the planes

$$\lambda\,(\phi\,x_1)\,\phi - p\,x_1,$$

$$\lambda\,(\phi\,x_2)\,\phi - p\,x_2.$$

The angle between these planes is

$$\frac{(p\,[\lambda\,(\phi\,x_1)\,\phi - p\,x_1]\,[\lambda\,(\phi\,x_2)\,\phi - p\,x_2])}{\{F\,[\lambda\,(\phi\,x_1)\,\phi - p\,x_1]\}\,\{F\,[\lambda\,(\phi\,x_2)\,\phi - p\,x_2]\}}.$$

Since $[F\,\phi]$ and $[\phi\,\phi]$ are zero this gives

$$\frac{-\lambda\,[(\phi\,x_1)\,(p\cdot\phi\cdot p\,x_2) + (\phi\,x_2)\,(p\cdot p\,x_1\cdot\phi) + (p:p\,x_1\cdot p\,x_2)]}{(F\,p\,x_1)\,(F\,p\,x_2)}$$

$$= \frac{-\lambda\,\{(\phi\,x_1)\,(\phi\,x_2) + (\phi\,x_2)\,(x_1\,\phi)\} + (p\,x_1\,x_2)}{(F\,p\,x_1)\,(F\,p\,x_2)}$$

$$= \frac{(p\,x_1\,x_2)}{(\phi\,x_1)\,(\phi\,x_2)}.$$

Hence the correlation changes x_1, x_2 into two planes ξ_1, ξ_2 such that

$$\overline{\xi_1\,\xi_2} = \overline{x_1\,x_2}.$$

In particular if $\lambda = 0$, the correlation between x and ξ is the null system determined by the complex p. The distance between any two points is therefore equal to the angle between their polar planes with respect to the complex p.

12. **Angle between two lines.** We define the angle between two lines r, s as such a function $\overline{r\,s}$ of their coordinates that, one of them being fixed and the angle constant, the other satisfies a linear relation (*i. e.* belongs to a linear complex) and for lines r, s, t of a plane pencil

$$\overline{r\,s} + \overline{s\,t} + \overline{t\,r} = 0.$$

By the same argument as for distance between two points we find

$$\overline{r\,s} = \frac{f_1\,(r,\,s)}{f_2\,(r,\,s)},$$

where

$$f_1\,(r,\,s) = -f_1\,(s,\,r)$$

and $f_2\,(r,\,s)$ factors into a linear function of r times the same linear function of s. Hence

$$\overline{r\,s} = \frac{(a\,r)\,(b\,s) - (a\,s)\,(b\,r)}{(c\,r)\,(c\,s)},$$

where a, b, c are matrices of two rows and $a\,b$ a dyadic setting up a correspondence between lines or complexes. The numerator of $\overline{r\,s}$ can be written in a different form. In fact

$$(A\,B\,r)\,(C\,D\,s) - (C\,D\,r)\,(A\,B\,s) =$$
$$\tfrac{1}{2}\{[ABC\cdot r\cdot D\,s] - [ABD\cdot r\cdot Cs] + [BCD\cdot r\cdot As] - [ACD\cdot r\cdot Bs]\},$$

as is seen by expanding the right hand member. The expression in the parentheses may be regarded as gotten by operating on the collineation (dyadic)

$$\tfrac{1}{2}\{[A\,B\,C]\,D - [A\,B\,D]\,C + [B\,C\,D]\,A - [A\,C\,D]\,\mathrm{B}\}$$

with r, s. For this collineation the linear invariant

$$\tfrac{1}{2}\,\{(A\,B\,C\,D) - (A\,B\,D\,C) + (B\,C\,D\,A) - (A\,C\,D\,B)\} = 0.$$

Such a collineation has sometimes been called normal. By summing we get

$$(a\,r)\,(b\,s) - (a\,s)\,(b\,r) = (\alpha\,r\cdot A\,s)$$

where $\alpha\,A$ is a collineation such that

$$(\alpha\,A) = 0.$$

Conversely if $\alpha\,A$ is any normal collineation

$$(\alpha\,r\cdot A\,r) = (\alpha\,A)\,\frac{(r\,r)}{2} = 0$$

r being any line or complex. Replacing r by $r + s$ we have

$$(\alpha\,r\cdot A\,s) + (\alpha\,s\cdot A\,r) = 0,$$

showing that $(\alpha\,r\cdot A\,s)$ changes sign with interchange of r and s and is hence of the type

$$(a\,r)\,(b\,s) - (a\,s)\,(b\,r).$$

We therefore have

$$\overline{r\,s} = \frac{(\alpha\,r\cdot A\,s)}{(c\,r)\,(c\,s)} \qquad . \quad . \quad . \quad . \quad (14)$$

It is to be noticed that this formula determines an angle between two complexes as well as between two lines. In particular the angle is zero if the complexes coincide.

The system of lines s making a zero angle with a line $r = [C\,D]$ may be constructed as follows. Let the correspondents of C and D through the collineation $\alpha\,A$ be

$$C' = (C\,\alpha)\,A$$
$$D' = (D\,\alpha)\,A.$$

Then s is determined by an equation

$$\begin{aligned}(\alpha\,r\cdot A\,s) &= (\alpha\cdot C\,D\cdot A\,s) = (\alpha\,D)\,(C\,A\,s) - (\alpha\,C)\,(D\,A\,s)\\ &= (D'\,C\,s) - (C'\,D\,s) = 0.\end{aligned}$$

In particular any line of the congruence cutting $D'C$ and $C'D$ has the required property. We may use instead of C, D any two points of the line. If then CD and $C'D'$ do not intersect this gives us an infinite number of congruences generating the complex to which s belongs.

13. For a general collineation αA these lines r, s making with each other zero angles have an interesting geometrical interpretation. It is well known that a general collineation whose linear invariant (αA) vanishes has a system of tetrahedra A, B, C, D such that each point is carried by the collineation into a point of the opposite face. Two opposite edges AB and CD of such a tetrahedron determine a zero angle. For in this case since C', D' are in the planes ABD and ABC, the lines $C'D$ and CD' cut AB.

Conversely if AB and CD are two non-intersecting lines making with each other a zero angle and those lines are not left entirely invariant by the collineation we construct a tetrahedron upon them as follows. Join A' and C', the correspondents of A, C through the collineation αA, to CD and AB respectively and let these planes determine on AB and CD respectively the points B and D. Then B will pass into a point B' such that AB' cuts CD (i. e. a point of ACD). Similarly for D. Thus, with the possible exception of fixed lines, the entire system of non-intersecting lines making with each other a zero angle consists of the opposite edges of these particular tetrahedra associated with the normal collineation αA.

If P, Q, R, S are any four points it is seen on expanding the right side that

$$(\alpha P)\,(A\,Q\,R\,S) = \tfrac{1}{2}\,[\alpha\,P\,Q\cdot A\,R\,S - \alpha\,P\,R\cdot A\,Q\,S + \alpha\,P\,S\cdot A\,Q\,R]$$

Hence if x is any point and ξ any plane $(\alpha\,x)\,(A\,\xi)$ is expressible as a sum of terms of the form $\alpha\,r\cdot A\,s$. Under any collineation leaving all angles invariant this last expression must be covariant. Hence the form $(\alpha\,x)\,(A\,\xi)$ must also be covariant.

Collineations leaving angle invariant must then leave the complex c invariant and the collineation αA fixed. We wish these angles to be invariant under the group of transformations that leave distance fixed. In that case c must coincide with p. There is a transformation of this group changing any distance $\overline{x\,y}$ into any equal distance $\overline{x\,y'}$. Since to x there can correspond through αA only one point y, this point must be fixed under all the collineations. Therefore to each point x corresponds the point F. Hence

$$\alpha A = \beta F$$

where β is a plane and F a point. Dual considerations show that β is fixed under all the collineations, i. e. coincides with ϕ. Hence by a proper choice of units we have

$$\overline{r\,s} = -\frac{(\phi\, r \cdot F\, s)}{(p\, r)\;(p\, s)} = \frac{(\phi\, s \cdot F\, r)}{(p\, r)\;(p\, s)} \quad . \quad . \quad . \quad (15)$$

The angle between two lines is zero if they cut a line through F in the plane ϕ. The angle is infinite if one of them belongs to the complex p and they are not cut by a line of the plane ϕ passing through the point F.

14. We have seen that

$$\overline{x\,y} = \lambda$$

sets up a correlation. To x and y correspond planes

$$\lambda(\phi\, x)\,\phi - p\, x,$$
$$\lambda(\phi\, y)\,\phi - p\, y.$$

To $x\,y$ corresponds the intersection which can be written

$$\lambda\,[\phi \cdot x\, y \cdot F] + (p\, x\, y)\, p - \tfrac{1}{2}\,(p\, p)\;[x\, y]$$

Hence to lines r and s correspond lines

$$\lambda\,[\phi\, r \cdot F] + (p\, r)\, p - r,$$
$$\lambda\,[\phi\, s \cdot F] + (p\, s)\, p - s.$$

The angle between these lines is

$$\frac{(\phi\;\{\lambda\,[\phi\, s \cdot F] + (p\, s)\, p - s\}\quad F\,\{\lambda\,[\phi\, r\, F] + (p\, r)\, p - r\})}{(p\,\{\lambda\,[\phi\, S\, F] + (p\, s)\, p - s\})\;(p\,\{\lambda\,[\phi\, r\, F] + (p\, r)\, p - r\})}$$
$$= \frac{(\phi\, s\, .\, F\, r)}{(p\, s)\,(p\, r)}.$$

Hence the angle between two lines is equal to that between the lines corresponding to them through the correlation.

$$\overline{x\,y} = \lambda$$

In particular when $\lambda = 0$ we see that the angle between two lines is equal to that between their polar lines with respect to the complex p.

15. **Distance from point to plane.** We wish to determine a function $\overline{A\,\alpha}$ of the coordinates of a point and plane such that if either is fixed the other satisfies a linear relation and such that

$$\overline{\mathrm{A}\,\alpha} = \overline{\mathrm{A}'\,\alpha'}$$

is a necessary and sufficient condition that A, α be transformable into A', α' by a motion leaving distance invariant. Such a function is

$$\overline{A\,\alpha} = \frac{(A\,\alpha)}{(\phi\,A)\,(F\,\alpha)} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Let α be a plane $B\,C\,D$. Then

$$\overline{A\,\alpha} = \frac{(A\,B\,C\,D)}{(\phi\,A)\,(F\,B\,C\,D)} = \frac{\frac{1}{2}\,(A\,B\,C\,D)\,(p\,p)}{(\phi\,A)\,(\phi\,p\cdot B\,C\,D)}.$$

This expression can be written

$$\frac{(p\,A\,B)\,(p\,C\,D) + (p\,A\,C)\,(p\,D\,B) + (p\,A\,D)\,(p\,B\,C)}{(\phi\,A)\{(\phi\,B)\,(p\,C\,D) + (\phi\,C)\,(p\,D\,B) + (\phi\,D)\,(p\,B\,C)\}}$$

$$= \frac{\overline{A\,B}\cdot\overline{C\,D} + \overline{A\,C}\cdot\overline{D\,B} + \overline{A\,D}\cdot\overline{B\,C}}{\overline{B\,C} + \overline{C\,D} + \overline{D\,B}} \qquad (17)$$

That $\overline{A\,\alpha}$ is invariant under the transformations leaving distance unchanged is shown by the last form. Conversely if

$$\overline{A\,\alpha} = \overline{A'\,\alpha'}$$

we take in α a triangle $B\,C\,D$ and in α' a corresponding triangle $B'C'D'$ such that

$$\overline{A'B'} = \overline{A\,B},\quad \overline{A'C'} = \overline{A\,C},\quad \overline{B'C'} = \overline{B\,C}$$
$$\overline{A'D'} = \overline{A\,D},\quad \overline{C'D'} = \overline{C\,D}.$$

Then the above equation shows that

$$\overline{B'\,D'} = \overline{B\,D}.$$

The two tetrahedra have all their edges equal and hence the one is transformable into the other.

This quantity $\overline{A\,\alpha}$ we call the distance from the point A to the plane α. It has many of the properties of euclidean distance from point to plane. Thus if the point lies in the plane (point not in ϕ and plane not through F) the distance is zero. If the plane is held fixed and the distance kept constant the point lies in a plane cutting α on ϕ. If the point is held fixed the locus of the plane is a point on the line joining the given point to F.

If $\qquad \overline{A\,\alpha} = \lambda$

the point corresponding to A (enveloped by α) is

$$B = \lambda\,(\phi\,A)\,F - A.$$

The distance from A to B is

$$\overline{A\,B} = \frac{(p\,A\,\{\lambda\,(\phi\,A)\,F - A\})}{-\,(\phi\,A)^2} = \lambda.$$

Thus $\overline{A\,B} = \overline{A\,\alpha}$. This shows that $\overline{A\,\alpha}$ is the distance, measured along $A\,F$, from A to the point of intersection of α with $A\,F$.

16. **Distance from point to line.** We define the distance from a point A to a line r as such a function of their coordinates that one of the quantities being fixed and the distance held constant, the other satisfies a linear relation and such that this distance is invariant under the transformations leaving distance between two points unchanged. Such a function is

$$\overline{A\,r} = \frac{(F\,A\,r)}{(\phi\,A)\,(p\,r)} \quad . \quad . \quad . \quad . \quad . \quad (18)$$

If r joins two points, B, C this can be written

$$\overline{A\,r} = \frac{(A\,B\,C\cdot\phi\,p)}{(A\,\phi)\,(B\,C\,p)} = \frac{(A\,\phi)\,(B\,C\,p) + (B\,\phi)\,(C\,A\,p) + (C\,\phi)\,(A\,B\,p)}{(A\,\phi)\,(p\,B\,C)}$$

Dividing numerator and denominator by $(A\,\phi)\,(B\,\phi)\,(C\,\phi)$, this becomes

$$\overline{A\,r} = \frac{\overline{B\,C} + \overline{C\,A} + A\,B}{\overline{B\,C}} \quad . \quad . \quad . \quad . \quad (19)$$

This expression shows that $\overline{A\,r}$ is invariant under the distance transformations. Conversely of

$$\overline{A\,r} = \overline{A'\,r'}$$

there is a transformation changing $\overline{A\,r}$ into $\overline{A'\,r'}$. For let B, C be two points of r. Take on r' two points B', C' such that

$$\overline{A\,B} = \overline{A'\,B'}$$

$$\overline{A\,C} = \overline{A'\,C'},$$

then

$$\overline{B\,C} = \overline{B'\,C'},$$

and a transformation of the kind desired can be obtained.

The distance from a point to a line is zero if the point lies on the line or if the plane of the line and the point pass through F (assuming that the point does not lie in ϕ and that the line does not belong to p). Since the order of A and r is immaterial in the formula for $\overline{A\,r}$ we write

$$\overline{A\,r} = \overline{r\,A}.$$

17. **Angle between line and plane.** Dual considerations give for the angle between a line r and a plane α, the expression

$$\overline{\alpha\,r} = \overline{r\,\alpha} = \frac{(\phi\,\alpha\,r)}{(F\,\alpha)\,(p\,r)} \quad . \quad . \quad . \quad . \qquad (20)$$

Let α be the plane at distance λ from A and r the line all points of which are at distance λ from s. Then

$$\alpha = \lambda\,(\phi\,A)\,\phi - p\,A,$$

$$r = \lambda\,[\phi\,S\,F] + (p\,s)\,p - s.$$

Hence

$$\overline{\alpha\,r} = \frac{(\phi\,\{\lambda\,(\phi\,A)\,\phi - p\,A\}\;\{\lambda[\phi\,s\,F] + (p\,s)\,p - s\})}{(F\,\{\lambda\,(\phi\,A)\,\phi - p\,A\})\,(p\,\{\lambda\,\phi\,s\,F + (p\,s)\,p - s\})}$$

$$= \frac{(-[p\,A]\,\{(p\,s)\,F - s\,\phi\})}{(\phi\,A)\,(p\,s)} = \frac{(p\,s)\,(\phi\,A) + (p\,A)\cdot(\phi\,s)}{(\phi\,A)\,(p\,s)}.$$

But $$[p\,A\cdot s\,\phi] = [p\,\{(A\,\phi)\,s - A\cdot s\,\phi\}],$$

and $$\{p\,[A\cdot\phi\,s]\} = (A\cdot s\cdot\phi\,p) = (A\,s\,F).$$

Hence $$\overline{\alpha\,r} = \frac{(F\,A\,s)}{(\phi\,A)\,(p\,s)} = \overline{A\,s}.$$

Therefore the angle between a line and plane is equal to the distance between the line and point corresponding to them through the distance correlation. In particular for $\lambda = 0$, we see that the distance between point and line is equal to the angle between their polar plane and point with respect to the complex p.

18. **Line Area of a triangle.** We define the area of a triangle $A\,B\,C$ as a function $\overline{A\,B\,C}$ of three points such that if the vertex is fixed and the base moved along its line, the area is proportional to the base. Hence if A is the vertex of the triangle and s the line on which the base $B\,C$ lies

$$\overline{A\,B\,C} = k\,\overline{B\,C}\cdot\overline{A\,s},$$

where k is a function of A and s. This gives on applying formulae (11) and (18) after replacing s by $[B\,C]$

$$\overline{A\,B\,C} = k\frac{(F\,A\,B\,C)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)}.$$

The areas of two triangles having the same vertex and base line are then proportional to the quantities

$$\frac{(F\,A\,B\,C)}{(\phi\,A)\,(\,(\phi\,B)\,(\phi\,C)}.$$

By a series of operations consisting of moving one side of the triangle along its line and keeping the opposite vertex fixed we can move the triangle into coincidence with any other having the same area. Under each of these operations the area is proportional to the above quantity. Hence any two areas are to each other as those quantities. Then by a proper choice of unit we have

$$\overline{A\,B\,C} = \frac{(F\,A\,B\,C)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)} \quad . \quad . \quad . \qquad (21)$$

Writing $F = (\phi\,p)$ we have

$$\overline{ABC} = \frac{(\phi\,p\cdot A\,B\,C)}{(\phi A)\,(\phi B)\,(\phi C)} = \frac{(\phi A)\,(pBC) + (\phi B)\,(p\,CA) + (\phi C)\,(pAB)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)}$$

$$= \overline{B\,C} + \overline{C\,A} + \overline{A\,B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

Thus the line area of a triangle is equal to its perimeter.

Dually we can find as the trihedral angle between three planes α, β, γ,

$$\overline{\alpha\beta\gamma} = \frac{(\phi\,\alpha\,\beta\,\gamma)}{(F\,\alpha)\,(F\,\beta)\,(F\,\gamma)}$$

$$= \overline{\alpha\gamma} + \overline{\beta\gamma} + \overline{\gamma\,\alpha} \quad . \quad . \quad . \qquad (23)$$

19. **Volume of a tetrahedron.** Similarly we define the volume of a tetrahedron $A\,B\,C\,D$ as such a function $\overline{A\,B\,C\,D}$ of the four points that given the vertex and plane of the base, the volume is proportional to the area of the base. From the definition we have

$$\overline{A\,B\,C\,D} = k\,\overline{B\,C\,D}\cdot\overline{A\,\alpha}$$

$$= \frac{k\,(F\,B\,C\,D)\cdot(\,A\,\alpha)}{(\phi\,B)\,(\phi\,C)\,(\phi\,D)\,(\phi\,A)\,(F\,\alpha)} = k\frac{(A\,B\,C\,D)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)\,(\phi\,D)},$$

where α represents the plane $B\,C\,D$, in which the base lies and k is a function of A and α. By a series of motions consisting of moving one triangle of the tetrahedron in its plane it is seen that the tetrahedron can be moved into any other having equal volume. These motions keep the volume constant and therefore k is an absolute constant. Hence choosing our unit so that $k = 1$, we have

$$\overline{A\,B\,C\,D} = \frac{(A\,B\,C\,D)}{(\phi\,A)\,(\phi\,B)\,(\phi\,C)\,(\phi\,D)} \quad . \quad . \qquad (24)$$

From the definition we have

$$\begin{aligned}\overline{A\,B\,C\,D} &= \overline{B\,C\,D}\cdot\overline{A\,\alpha} = \overline{B\,C\,D}\cdot\overline{A\,(B\,C\,D)}\\ &= (\overline{B\,C}+\overline{C\,D}+\overline{D\,B})\left(\frac{(\overline{A\,B}\cdot\overline{C\,D}+\overline{A\,C}\cdot\overline{D\,B}+\overline{A\,D}\cdot\overline{B\,C})}{\overline{B\,C}+\overline{C\,D}+\overline{D\,B}}\right)\\ &= \overline{A\,B}\cdot\overline{C\,D}+\overline{A\,C}\cdot\overline{D\,B}+\overline{A\,D}\cdot\overline{B\,C} \quad . \quad . \quad . \qquad (25)\end{aligned}$$

From (24) we see that if the vertex A lies in the plane $A\,B\,C$ the volume is zero. Hence applying this to (25) we have

$$\overline{A\,B}\cdot\overline{C\,D}+\overline{A\,C}\cdot\overline{D\,B}+\overline{A\,D}\cdot\overline{B\,C} = 0$$

as a relation connecting four points lying in a plane. This relation is seen to be identical with the relation connecting the Plücker coordinates of a line. From this a theory of plane quadrilaterals could be built up.

20. **Summary.** We have defined a bilinear function of any two spaces in three dimensions. In case one of these spaces is a point we call this function a distance otherwise an angle. We have also defined certain areas determined by three elements and volumes determined by four. These functions are all invariant under a six parametered group of collineations projectively equivalent to the group of collineations leaving euclidean volume invariant. Under the correlation

$$\overline{x\,y} = \text{const.}$$

each of these functions is equal to the dual function of the transformed elements. The expressions for these functions are

$$\overline{A\,B} = \frac{(p\,A\,B)}{(\phi\,A)\,(\phi\,B)} \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

$$\overline{\alpha\,\beta} = \frac{(p\,\alpha\,\beta)}{(F\,\alpha)\,(F\,\beta)} \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

$$\overline{r\,s} = \frac{(\phi\, S \cdot F\, r)}{(p\, s)\ (p\, r)} \quad . \quad . \quad . \quad . \quad . \qquad (15)$$

$$\overline{A\,\alpha} = \frac{(A\, \alpha)}{(\phi\, A)\ (F\, \alpha)} \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

$$\overline{A\,r} = \frac{(F\, A\, r)}{(\phi\, A)\ (p\, r)} \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

$$\overline{r\,\alpha} = \frac{(\phi\, r\, \alpha)}{(p\, r)\ (F\, \alpha)} \quad . \quad . \quad . \quad . \quad . \qquad (20)$$

$$\overline{A\,B\,C} = \frac{(F\, A\, B\, C)}{(\phi\, A)\ (\phi\, B)\ (\phi\, C)} \quad . \quad . \quad . \quad . \qquad (21)$$

$$\overline{\alpha\,\beta\,\gamma} = \frac{(\phi\, \alpha\, \beta\, \gamma)}{(F\, \alpha)\ (F\, \beta)\ (F\, \gamma)} \quad . \quad . \quad . \quad . \qquad (23)$$

$$\overline{A\,B\,C\,D} = \frac{(A\, B\, C\, D)}{(\phi\, A)\ (\phi\, B)\ (\phi\, C)\ (\phi\, D)} \quad . \quad . \quad . \qquad (24)$$

$$\overline{\alpha\,\beta\,\gamma\,\delta} = \frac{(\alpha\, \beta\, \gamma\, \delta)}{(F\, \alpha)\ (F\, \beta)\ (F\, \gamma)\ (F\, \delta)} \quad . \quad . \quad . \qquad (25)$$

21. **Tetrahedron.** The angles of a triangle will now be expressed in terms of the sides. For the angle $C\,A\,B$ of the triangle $A\,B\,C$ we have

$$\text{Angle } C\,A\,B = \overline{C\,A \cdot A\,B} = \frac{(\phi\, A\, B \cdot F\, C\, A)}{(p\, A\, B)\ (p\, C\, A)}$$

$$= \frac{(\phi\, A)\ (F\, A\, B\, C)}{(p\, A\, B)\ (p\, C\, A)}.$$

Replacing F by $[\phi\, p]$ and applying (3) we have

$$\overline{C\,A,\, A\,B} = \frac{(\phi\, A)\ (\phi\, p \cdot A\, B\, C)}{(p\, A\, B)\ (p\, C\, A)}$$

$$= \frac{(\phi\, A)\ \{(\phi\, A)\ (p\, B\, C) + (\phi\, B)\ (p\, C\, A) + (\phi\, C)\ (p\, A\, B)\}}{(p\, A\, B)\ (p\, C\, A)}.$$

Dividing numerator by $(\phi\, A)^2\ (\phi\, B)\ (\phi\, C)$, this becomes

$$\overline{C\,A,\, A\,B} = \frac{\overline{B\,C} + \overline{C\,A} + \overline{A\,B}}{\overline{A\,B} \cdot \overline{C\,A}}$$

If we use A, B, C for the angles and a, b, c for the sides opposite this becomes

$$A = \frac{a+b+c}{b\,c} \quad . \quad . \quad . \quad . \qquad (26)$$

Similarly we have

$$a = \frac{A+B+C}{B\,C} \quad . \quad . \quad . \quad . \qquad (27)$$

In the tetrahedron if α, β, γ, δ are the planes opposite the vertices A, B, C, D we have for the angle

$$\overline{\alpha\beta} = \frac{(p\cdot BCD\cdot CDA)}{(FBCD)\,(FCDA)} = \frac{(pCD)\,(BCDA)}{(FBCD)\,(FCDA)}$$

$$= \frac{\overline{CD}\;\overline{BCDA}}{\overline{BCD}\;\overline{CDA}}.$$

This gives for the volume

$$\overline{BCDA} = \frac{\overline{BCD}\;\overline{CDA}\;\overline{\alpha\beta}}{\overline{CD}} \quad . \quad . \qquad (28)$$

That is the volume of a tetrahedron is equal to the product of the areas of two faces and the dihedral angle between them divided by the length of the common edge.

The trihedral angle $\overline{\alpha\beta\gamma}$ is given by

$$\overline{\alpha\beta\gamma} = \overline{\alpha\beta} + \overline{\beta\gamma} + \overline{\gamma\alpha}$$

$$= \frac{\overline{CD}\cdot\overline{BCDA}}{\overline{BCD}\cdot\overline{CDA}} + \frac{\overline{DA}\cdot\overline{CDAB}}{\overline{CDA}\cdot\overline{DAB}} + \frac{\overline{DB}\cdot\overline{ADBC}}{\overline{ADB}\cdot\overline{CDB}}$$

$$= \overline{BCDA}\left(\frac{\overline{DAB}\cdot\overline{CD} + \overline{BCD}\cdot\overline{DA} + \overline{CDA}\cdot\overline{DB}}{\overline{BCD}\cdot\overline{CDA}\cdot\overline{DAB}}\right). \qquad (29)$$

This formula solved for $\overline{BCDA}$ will also express the volume in terms of the trihedral angle and the three face triangles and three edges which meet at its vertex.

The volume can also be expressed in many other forms.

Linear distance in hyperspace.

22. The argument by which we derived the formula for the distance between two points in three dimensions applies without change to higher dimensions. The formula for distance is then always

$$\overline{A\,B} = \frac{(q\,A\,B)}{(\phi\,A)\;(\phi\,B)},$$

where q is a complex matrix of order $n-2$ and ϕ a hyperplane. Similarly the angle between two hyperplanes is

$$\overline{\alpha\,\beta} = \frac{(p\,\alpha\,\beta)}{(F\,\alpha)\;(F\,\beta)},$$

where p is a complex matrix of order two and F a point. We wish these quantities to be invariant under the same group of collineations. This will happen if ϕ and q are determined by F and p and conversely. We shall therefore consider the system of complexes determined by a point F and a complex p of the second order. The details of this discussion depend somewhat on whether the space is of even or odd order. We consequently consider these cases separately.

23. **Space of order** $n = 2\,m$. The progressive products of a complex p with itself give a system of complexes $[p\,p]$, $[p\,p\,p]$ etc. we shall denote these by the symbols p^2, p^3 etc. In the present case p^m is represented by a sum of determinants of order n and hence is a scalar. We assume that this quantity is not zero. Such for example is the case if

$$p = A_1\,A_2 + A_3\,A_4 + \;.\;.\;.\;.\;.\; A_{2m-1}\;A_{2m}$$

and the points A_i do not lie in a hyperplane. For then

$$p^m \;=\; m!\;(A_1\,A_2\;.\;.\;.\;.\;.\;A_{2m}).$$

Since p^m is not zero none of the lower powers are zero.

We take as a fundamental system the quantities

$$F,\;p,\;F\,p,\;p^2\;.\;.\;.\;.\;F\,p^{m-1},$$

consisting of the powers of p and those powers multiplied by F. We shall find that this system forms a group under progressive and regressive multiplication, in the sense that the product of any two is either zero or a numerical multiple of a third in the system.

To form products it is sufficient to recall that p is a sum of products of two points and hence in *linear* (distributive) operations behaves like a simple product of two points. Furthermore to multiply regressively

R by a product of points S we take from S all combinations D of points such that D is complimentary to R, arrange the others in a product C such that $S = C\,D$, and form the sum $\Sigma\,(R\,D)\,C$. To obtain the product

$$[p^2 \cdot p^{m-1}]$$

by resolving the second factor, we must take the sum of products, of p^2 by all but two letters of any term of the second factor times the product of those two. Those letters will occur in a combination p^2 and this combination may be selected in $\binom{m-1}{2}$ ways. Hence

$$[p^2 \cdot p^{m-1}] = \binom{m-1}{2}[p^{m-1} \cdot p^2] - \left(2\binom{m-1}{2} - (m-1\right)p^m \cdot p,$$

the second term being subtracted because in

$$\binom{m-1}{2}[p^{m-1} \cdot p^2]$$

occur $2\binom{m-1}{2}$ terms of the form $p^m \cdot p$, whereas there should be $m-1$ in the expansion of $[p^2 \cdot p^{m-1}]$. Simplifying the above expression we get, since $\quad [p^2 \cdot p^{m-1}] = [p^{m-1} \cdot p^2]$,

$$[p^{m-1} \cdot p^2] = 2\,\frac{m-1}{m}\,p^m \cdot p.$$

Similarly

$$\begin{aligned}[p^{m-1} \cdot p^r] &= \binom{r}{2}[p^{m-1} \cdot p^2 \cdot p^{r-2}] - [2\binom{r}{2} - r]\,p^m\,p^{r-1} \\ &= r\,\frac{(m-r+1)}{m}\,p^m \cdot p^{r-1}.\end{aligned}$$

Since p^m is a scalar, we may solve this last equation for p^{r-1}. Changing r into $r+1$ in the result we have,

$$p^r = \frac{m\,[p^{m-1}\,p^{r+1}]}{p^m\,(r+1)\,(m-r)},$$

the equation holding for $r = 0$ if we take $p^0 = 1$. Thus we have an expression for p^r in terms of p^{r+1}. Expressing p^{r+1} in terms of p^{r+2}, etc., we have finally

$$p^r = \left(\frac{m\,p^{m-1}}{p^m}\right)^{m-r} \frac{p^m}{[(r+1)\,(r+2)\,\ldots\,m]\,(m-r)!}.$$

If we choose the magnitude of p such that

$$p^m = m!$$

and let

$$\frac{p^{m-1}}{(m-1)!} = q,$$

this equation may be written

$$\frac{p^r}{r!} = \frac{q^{m-r}}{(m-r)!} \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

where $r = 0, 1, \ldots . m$.

Again we have

$$[F p^m] = m\,[p^{m-1} F \cdot p]$$

and

$$[p^{m-1} F \cdot p^r] = r\,[(p^{m-1} F \cdot p)\, p^{r-1}].$$

Hence

$$[p^{m-1} \cdot F p^r] = F\,[p^{m-1} p^r] - [p^{m-1} F \cdot p^r] = \frac{r\,(m-r)}{m}\, p^m\, [F p^{r-1}].$$

Solving this for $[F p^{r-1}]$ and changing r into $r + 1$, we have

$$[F p^r] = \frac{m\,[p^{m-1} \cdot F p^{r+1}],}{p^m\,(m-r-1)\,(r+1)},$$

a formula holding for $r = 0$. By continued application of this formula we finally get

$$\frac{[F p^r)}{r!} = \frac{[q^{m-r-1} F p^{m-1}]}{(m-r-1)!\,(m-1)!}.$$

Let

$$\frac{[F p^{m-1}]}{(m-1)!} = \phi.$$

Then

$$\frac{[F p^r]}{r!} = \frac{\phi\, q^{m-r-1}}{(m-r-1)!} \quad . \quad . \quad . \quad . \qquad (31)$$

where $r = 0, 1, 2, \ldots m - 1$.

24. **Space of order** $n = 2m + 1$. In this case p^m is of order $n - 1$ and hence represents a hyperplane. Since the product p^m is progressive this product must contain p (i. e., p can be expressed as $\Sigma\, \lambda_{ik}\, [A_i\, A_k]$, the points A_i being contained in ϕ). Hence,

$$[p \cdot p^m] = 0.$$

We assume that

$$[F\,p^m] \neq 0.$$

Then p^r and $[F\,p^r]$, $r \leqq m$, are not zero.

Since $[p\cdot p^m] = o$, there can be no terms in the expansion of $[p\cdot F\,p^m]$ which have F outside the parenthesis. Hence

$$p\,(F\,p^m) = m\,(F\,p^m)\,p + \binom{m}{2}[F\,p^{m-1}\cdot p^2] - 2\binom{m}{2}(F\,p^m)\,p.$$

Consequently

$$[F\,p^{m-1}.\,p^2] = 2\,\frac{m-1}{m}\,(F\,p^m)\,p.$$

Similarly

$$[F\,p^{m+1}\cdot p^r] = \binom{r}{2}[F\,p^{m-1}\cdot p^2\cdot p^{r-2}] - [2\binom{r}{2} - r]\,(F\,p^m)\,p^{r-1}$$

$$= \frac{r\,(m-r+1)}{m}\,(F\,p^m)\,p^{r-1}$$

Solving this for p^{r-1} and changing r into $r+1$, we obtain

$$p^r = \frac{m\,[F\,p^{m-1}\cdot p^{r+1}]}{(F\,p^m)\,(r+1)\,(m-r)}.$$

Repeated use of this formula gives finally

$$\frac{p^r}{r!} = \left(\frac{m\,F\,p^{m-1}}{F\,p^m}\right)^{m-r}\frac{p^m}{m!\,(m-r)!}.$$

If we choose the magnitudes of F and p such that

$$(F\,p^m) = m! \quad . \quad . \quad . \quad . \quad . \qquad (A)$$

and let

$$\frac{F\,p^{m-1}}{(m-1)} = q. \quad . \quad . \quad . \quad . \quad . \qquad (B)$$

$$\frac{p^m}{m!} = \phi \quad . \quad . \quad . \quad . \quad . \qquad (C)$$

we have

$$\frac{p^r}{r!} = \frac{[q^{m-r}\,\phi]}{(m-r)!} \quad . \quad . \quad . \quad . \quad . \qquad (32)$$

where $r = 0, 1, \ldots m$.

Letting $r = 0$ in (32) we have

$$m! = q^m\cdot\phi \quad . \quad . \quad . \quad . \quad . \quad . \qquad (A')$$

Hence q^m is not zero. Now q^m is a point, and since q contains F as a factor, must be the point F. Also

$$1 = [F\,\phi].$$

Consequently

$$\frac{q^m}{m!} = F \quad . \quad . \quad . \quad . \quad . \quad . \qquad (C')$$

again from (32) we have

$$\frac{[\phi\, q^{m-1}]}{(m-1)!} = p \quad . \quad . \quad . \quad . \quad . \qquad (B')$$

The equations (A), (B), (C) and (A'), (B'), (C') show that F, p are related to ϕ, q in the same way that the latter are to the former. Hence

$$\frac{q^r}{r!} = \frac{[p^{m-r} \cdot F]}{(m-r)!} \quad . \quad . \quad . \quad . \quad . \qquad (33)$$

where $r = 0, 1, 2 \;.\;.\;.\; m$.

The formulae (30), (31), (32), (33) show that the system of quantities p^r, $F\,p^r$ is generated in the same way from F, p or from ϕ, q. If the product of two of these quantities is progressive the factors can be associated and the result is either zero or equal to a third. If the product is regressive we replace p^r and $F\,p^r$ by their expressions in terms of ϕ and q. The product in this form is represented by a sum of matrices (in hyperplane coordinates) having a smaller number of rows than columns. The factors can therefore be commuted and associated giving a result which is number times a quantity of the form q^r or $\phi\,q^r$. Hence the product of any two quantities of the fundamental system is a numerical multiple of a third.

Let

$$\left.\begin{aligned} S_{2r} &= \frac{p^r}{r!} \\ S_{2r+1} &= \frac{F\,p^r}{r!} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

where $r = 0, 1, 2, \;.\;.\;.\; m$. Dually we have the quantities σ_i such that,

$$\left.\begin{aligned} \sigma_{2r} &= \frac{q^r}{r!} \\ \sigma_{2r+1} &= \frac{\phi q^r}{r!} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (35)$$

where $r = 0, 1, 2, \ldots m$. Equations (30), (31), (32), (33) show that

$$\sigma_i = S_{n-i} \quad . \quad . \quad . \quad . \quad . \quad (36).$$

25. **Distance and angle.** The distance between two points A, B is

$$\overline{A\,B} = \frac{(q\,A\,B)}{(\phi\,A)\,(\phi\,B)}.$$

Similarly we define the angle between any two spaces R, T of the same order r by the equation

$$\overline{R\,T} = \frac{(q \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T)}{(\sigma_r\,R)\,(\sigma_r\,T)}.$$

σ_{r-1} being the complex which multiplied by R and T respectively give points. This expression can be put into two other forms which we shall now obtain. We consider three cases depending on the form of σ_{r+1}.

(1) If

$$\sigma_{r+1} = \frac{p^k}{k!},$$

we have

$$[\sigma_{r+1}\,R] = \frac{[p^k \cdot R]}{k!} = \frac{[p \cdot p^{k+1}\,R]}{(k-1)!} = [p \cdot \sigma_{r-1}\,R].$$

Then

$$(p \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T) = (\sigma_{r+1}\,R \cdot \sigma_{r-1}\,T).$$

(2). If

$$\sigma_{r+1} = \frac{[F\,p^k]}{k!},$$

then

$$[\sigma_{r+1}\,R] = \frac{(p^k\,R)\,F}{k!} - \frac{[p \cdot F\,p^{k-1}R]}{(k-1)!}.$$

Since in this case σ_{r-1} also contains F, we have

$$(p \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T) = -(\sigma_{r+1}\,R \cdot \sigma_{r-1}\,T).$$

In both of the preceding cases

$$(p \cdot \sigma_{r-1}\,R \cdot \sigma_{r-1}\,T) = (-1)^{r+1}\,(\sigma_{r+1}\,R \cdot \sigma_{r-1}\,T).$$

(3) If σ_{r+1} is of the form $\frac{q^k}{k!}$ or $\frac{[\phi q^k]}{k!}$ by the dual of the preced-

ing reasoning we have, since the sign must be positive in the first and negative in the second case

$$\begin{aligned}(\sigma_{r+1}\, R \cdot \sigma_{r-1}\, T) &= (-\,1)^{n-r-1}\, (\sigma_{r+1}\, R \cdot q \cdot \sigma_{r+1}\, T) \\ &= (-\,1)^{r+1}\, (q \cdot \sigma_{r+1}\, R \cdot \sigma_{r+1}\, T).\end{aligned}$$

For every case the following equation holds:

$$\overline{R\,T} = \frac{(p \cdot \sigma_{r-1} R \cdot \sigma_{r-1}\, T)}{(\sigma_r\, R)\; (\sigma_r\, T)} = \frac{(q \cdot \sigma_{r+1} R \cdot \sigma_{r+1}\, T)}{(\sigma_r\, R)\; (\sigma_r\, T)} = (-\,1)^{r+1} \frac{(\sigma_{r+1}\, R \cdot \sigma_{r-1}\, T)}{(\sigma_r\, R)\; (\sigma_r\, T)}.$$

It is evident from the definition that

$$\overline{R\,T} = -\,\overline{T\,R}.$$

This together with the linearity of the expression, the factored form and symmetry of the denominator, shows that three spaces R, R', R'' of a pencil determine angles such that

$$\overline{R\,R'} + \overline{R'\,R''} + \overline{R''\,R} = 0.$$

To prove this directly it is only necessary to place

$$R'' = \lambda\, R + \mu\, R'$$

in the expression for the above sum and clear of fractions.

26. **Distance and angle in a section of hyperspace.** A space R of our space of order n intersects the complexes S_i of the fundamental system in a set of complexes. For spaces contained in R we can define distance and angle relative to these last complexes. We wish now to show the relation between those invariants and the corresponding invariants relative to the complexes S_i.

First consider the section made by a hyperplane α. This determines with the complex p a point

$$F_1 = [\alpha\, p],$$

and with the complex $[F\, p]$, a complex

$$p_1 = [\alpha \cdot F\, p].$$

We can write this last expression in the form

$$p_1 = (\alpha\, F)\, p - [\alpha \cdot p \cdot F].$$

If we multiply this by itself r times, since the last term is a line, this last term cannot appear more than once as a factor of any term of the result. Hence

$$
\begin{aligned}
p_1{}^r &= (\alpha\, F)^r\, p^r - r\,(\alpha\, F)^{r-1}\,[\alpha\cdot p\cdot F\cdot p^{r-1}]. \\
&= (\alpha\, F)^{r-1}\,\{(\alpha\, F)\, p^r - r\,[\alpha\cdot p\cdot F\cdot p^{r-1}]\}. \\
&= (\alpha\, F)^{r-1}\,[\alpha\cdot F\, p^r],
\end{aligned}
$$

provided that $2\,r + 1 \leqq n$. Multiplying the first of these values of $p_1{}^r$ by $F_1 = [\alpha\, p]$, since $[\alpha\, p]$ is already a factor of the second term, we get

$$
[F_1\, p_1{}^r] = (\alpha\, F)^r\,[\alpha\, p\cdot p^r] = \frac{(\alpha\, F)^r\,[\alpha\cdot p^{r+1}]}{r+1},
$$

provided that $2\,r + 2 \leqq n$. Dividing the above expressions for $p_1{}^r$ and $[F_1\, p_1{}^r]$ by $r!$, we get

$$
\left.
\begin{aligned}
\frac{p_1{}^r}{r!} &= \frac{(\alpha\, F)^{r-1}\,[\alpha\cdot F\, p^r]}{r!} \\
\frac{[F_1\, p_1{}^r]}{r!} &= \frac{(\alpha\, F)^r\,[\alpha\cdot p^{r+1}]}{(r+1)!}
\end{aligned}
\right\} \quad . \quad . \quad . \qquad (37)
$$

these expressions being valid if the order of the left side is equal to or less than that of a hyperplane.

We next find the intersection of a second hyperplane β with the system of complexes $p_1{}^r$, $[F_1\, p_1{}^r]$. Let

$$
F_2 = [\beta\, p_1] = [\beta\, \alpha\cdot F\, p]
$$

$$
p_2 = \frac{[\beta\cdot F_1\ p_1]}{(\alpha\, F)} = \frac{[\beta\ \alpha\cdot p^2]}{2!}.
$$

By the same argument as before we get

$$
\frac{p_2{}^r}{r!} = \frac{(\beta\, F_1)^{r-1}[\,\beta\cdot F_1\, p_1{}^r]}{(\alpha\, F)^r\, r!}
$$

$$
\frac{[F_2\, p_2{}^r]}{r!} = \frac{(\beta\, F_1)^r\,[\beta\cdot p_1{}^{r+1}]}{(\alpha\, F)^r\,(r+1)!}.
$$

Using the values of p_1 and $[F_1\, p_1{}^r]$ in (37) we have

$$
\frac{p_2{}^r}{r!} = \frac{(\beta\,\alpha\, p)^{r-1}\,[\beta\,\alpha\cdot p^{r+1}]}{(r+1)!},
$$

$$
\frac{[F_2\, p_2{}^r]}{r!} = (\beta\ \alpha\ p)^r\,\frac{[\beta\,\alpha\cdot F\, p^{r+1}]}{r!},
$$

these expressions being valid if the order of the left member is equal to or less than that of $[\beta\ \alpha]$. Similarly we obtain the intersection of this system with a hyperplane γ, etc. We thus get finally.

$$F_\lambda = [R\ S_{\lambda+1}],$$
$$p_\lambda = [R\ S_{\lambda+2}].$$

as point and complex in a space R of order $n - \lambda$. For these we have the equations

$$\left.\begin{aligned} R_{2r} &= \frac{p_\lambda{}^r}{r!} = (R\ S_\lambda)^{r-1}\,[R\cdot S_{2r+\lambda}] \\ R_{2r+1} &= \frac{[F^n p_\lambda{}^r]}{r!} = (R\ S_\lambda)^r[R\cdot S_{2r+\lambda+1}] \end{aligned}\right\} \quad . \qquad (38)$$

these expressions being valid for values of r such that the orders of the left members are equal to or less than $n - \lambda$.

If $(R\ S_\lambda) \neq 0$ we choose the magnitude of R such that

$$(R\ S_\lambda) = 1.$$

Then the above equations become

$$R_m = [R\cdot S_{m+\lambda}] \quad . \quad . \quad . \quad . \qquad (39)$$

We consequently have

$$\frac{[R_{n-\lambda-2}\cdot A\ B]\,R}{[R_{n-\lambda-1}\ A][R_{n-\lambda-1}\ B]} = \frac{[R\ S_{n-r}\cdot A\ B]\ R}{[R\ S_{n-1}\cdot A]\ [R\ S_{n-1}\cdot B]}$$
$$= \frac{[A\ B\cdot R\ S_{n-2}]\,R}{[A\cdot R\ S_{n-1}]\ [B\cdot R\ S_{n-1}]} = \frac{R\ (A\ B\ S_{n-2})\ R}{R\ (A\ S_{n-1})\ R\ (B\ S_{n-1})}$$

provided that $A\ B$ is contained in R.[12]

Hence we have

$$\overline{A\ B} = \frac{(S_{n-2}\cdot A\ B)}{(S_{n-1}\ A)\ (S_{n-1}\ B)} = \frac{[R_{n-\lambda-2}\ A\ B]\ R}{[R_{n-\lambda-1}\ A]\ [R_{n-\lambda-1}\ B]}.$$

We may consider R as the unit quantity in the space R. Then the right side of the above equation is the expression for distance relative to the system of complexes in R. Thus whether we take distance in R relative to the fundamental system of complexes S_i or relative to the sections R_i in R, the result is the same. Similar relations of the angles between other spaces in R relative to S_i and R_i can be shown.

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[12] Cf. Grassmann, Gesammelte Werke, Vol. I, theil 2, page 91.